10.9 and 10.10: Bounds and Applications of Taylor Series

Taylor's Formula. If f has derivatives of all orders in an open interval I containing a, then $\forall n \in \mathbb{N}$ and $\forall x \in I$,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x),$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \text{for some } c \text{ between } a \text{ and } x.$$

We call $R_n(x)$ the remainder of order n or the error term for the approximation of f by $P_n(x)$ over I. If $R_n(x) \to 0$ as $n \to \infty$ $\forall x \in I$, we say that the Taylor series generated by f at $x = 1$ converges to f on I, and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

The Remainder Estimation Theorem. If there exists an $M \in (0, \infty)$ such that $|f^{(n+1)}(t)| \leq M$ for all t between x and a, including x and a, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \leq M \frac{|x-a|^{n+1}}{(n+1)!}.$$

Furthermore, if this $|f^{(n+1)}(t)| \leq M$ $\forall n \in \mathbb{N}$ and all t between x and a, then the Taylor series converges to $f(x)$ on I.

Examples.

1. Show that the Maclaurin series of $f(x) = e^x$ converges to $f(x)$ for all $x \in \mathbb{R}$.
2. Show that the Maclaurin series for $\sin x$ converges for all $x \in \mathbb{R}$.
3. Show that the Maclaurin series for $\cos x$ converges for all $x \in \mathbb{R}$.
4. Find the first few terms of the Maclaurin series for $\frac{1}{3}(2x + x\cos x)$.
5. Find the first few terms of the Maclaurin series for $e^x \cos x$.
6. For what values of x can we replace $\sin x$ by $x - (x^3/3!)$ with an error of no greater than 3×10^{-4}?

The Binomial Series.
For $-1 < x < 1$,

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k,$$

where we define

$$\binom{m}{1} = m, \quad \binom{m}{2} = \frac{m(m-1)}{2!},$$

and

$$\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!} \quad \text{for } k \geq 3.$$

Notice that, if $m \in \mathbb{N}$ and $0 \leq k \leq m$ is an integer, then the binomial coefficient $\binom{m}{k}$ is given by

$$\binom{m}{k} = \frac{m!}{k!(m-k)!}.$$

Examples.

1. Find $\binom{-1}{1}$ and $\binom{-1}{2}$.
2. Represent $\sqrt{1+x}$ as a binomial series and use the Alternating Series Estimation Theorem to get bounds for the Taylor polynomial of order 2.
3. You can squeeze even more from the binomial series with $m = \frac{1}{2}$ to get bounds for $\sqrt{1-x^2}$, $\sqrt{1-\frac{1}{x}}$ and many other Taylor series.

Evaluating Nonelementary Integrals and Indeterminate Forms. Examples.

1. Express $\int \sin x^2 dx$ as a power series.
2. Estimate $\int_0^1 \sin x^2 dx$ with an error less than 0.001.
3. Integrate term-by-term the power series for $\frac{1}{1+x^2}$ to show that $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ for $|x| \leq 1$. By setting $x = 1$ in this power series expansion we get a series representation of $\frac{\pi}{4}$ called **Leibniz's formula.**
4. Evaluate $\lim_{x \to 1} \frac{\ln x}{x-1}$.
5. Find $\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$.